$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + \lambda f(x_1, y_1, z_1) + \mu F(x_2, y_2, z_2),$$

where λ and μ are constants, shall all vanish. This gives us six equations, of which these are types:

$$2(x_1 - x_2) + \lambda \frac{\partial f}{\partial x_1} = 0,$$

$$-2(x_1 - x_2) + \mu \frac{\partial F}{\partial x_2} = 0.$$

From these, we have at once

$$\frac{\partial f}{\partial x_1} : \frac{\partial f}{\partial y_1} : \frac{\partial f}{\partial z_1} = x_1 - x_2 : y_1 - y_2 : z_1 - z_2.$$

But $\partial f/\partial x_1$, $\partial f/\partial y$, $\partial f/\partial z_1$, are proportional to the direction cosines of the normal to $f(x_1, y_1, z_1) = 0$ at (x_1, y_1, z_1). And $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$, are proportional to the direction cosines of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2). Hence this line coincides with the normal. Similarly it is normal to $F(x_1, y_1, z_1) = 0$. Hence, the shortest distance must be perpendicular to both surfaces, *but not necessarily conversely.*

372. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find the condition that the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{a^2}{x^2}\right) y = 0$$

should have one solution expressible in integral powers of x; and show that if this condition is satisfied, every other solution of the equation possesses a logarithmic infinity at the origin.

SOLUTION BY ALBERT N. NAUER, St. Louis, Mo.

Let $y = vx^{-\frac{1}{2}}$. Then finding the first and second derivatives of y with respect to x and substituting these values in the equation, we have

$$\frac{d^2v}{dx^2} - v = \left(\frac{a^2 - \frac{1}{4}}{x^2}\right) v.$$

This equation is of the form

$$\frac{d^2u}{dx^2} - \alpha^2 u = \frac{m(m+1)}{x^2} u,$$

where $m(m+1) = a^2 - \frac{1}{4}$, or $m = -\frac{1}{2} \pm a$ and $\alpha = 1$. This equation has the three following known solutions:

$$(1)\quad u = x^{-m}\left(1 - \frac{1}{m - \frac{1}{2}}\frac{\alpha^2x^2}{2^2} + \frac{1}{(m - \frac{1}{2})(m - \frac{3}{2})}\frac{\alpha^4x^4}{2!\,2^4} - \frac{1}{(m - \frac{1}{2})(m - \frac{3}{2})(m - \frac{5}{2})}\frac{\alpha^6x^6}{3!\,2^6} + \cdots\right),$$

$$(2)\quad u = e^{\alpha x}x^{-m}\left(1 - \frac{m}{m}\alpha x + \frac{m(m-1)}{m(m-\frac{1}{2})}\frac{\alpha^2x^2}{2!} + \frac{m(m-1)(m-2)}{m(m-\frac{1}{2})(m-1)}\frac{\alpha^3x^3}{3!} - \cdots\right),$$

$$(3)\quad u = e^{-\alpha x}x^{-m}\left(1 + \frac{m}{m}\alpha x + \frac{m(m-1)}{m(m-\frac{1}{2})}\frac{\alpha^2x^2}{2!} - \frac{m(m-1)(m-2)}{m(m-\frac{1}{2})(m-1)}\frac{\alpha^3x^3}{3!} + \cdots\right).$$

Solution (2) is a finite primitive solution when m is an integer. When m is not an integer or zero, solution (1) gives an infinite series in powers of x. To have integral powers of x, m must equal $-(2n+1)/2$ when n is any integer or zero; also in the original equation $\pm a$ must equal $(2n+3)/2$.

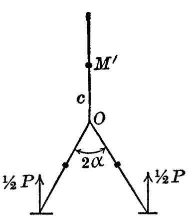
MECHANICS.

293. Proposed by B. F. FINKEL, Drury College.

A man of weight w stands on smooth ice; prove that if, when he gradually parts his legs, kept straight, with his feet in contact with the ice, the pressure of his feet on the ice be constant, his head will descend with uniform acceleration; and that, if f be the acceleration of his head, when his feet exert no pressure on the ice, their pressure on the ice, if f' were the acceleration of his head, would be equal to $\frac{f-f'}{f}w$. Walton's *Problems in Theoretical Mechanics*, p. 662.

Solution by E. B. Wilson, Massachusetts Institute of Technology.

We may analyze the man's total mass, M, into M'', the mass of the legs, and M', the remaining mass. The forces acting are W down and P up. Let 2α be the angle between the legs, l their length, $a\cos\alpha$ the distance of their center of gravity below O. Let c be the distance of the center of gravity of M' above O. The head falls through the distance $l(1-\cos\alpha)$. The center of gravity of the whole mass is at a height



$$h = \frac{M'(c + l\cos\alpha) + M''(l-a)\cos\alpha}{M = M' + M''}$$

above the ice. Hence the downward acceleration of the head is

$$f' = -l\frac{d^2\cos\alpha}{dt^2}.$$

The downward acceleration of his C.G. is

$$-\frac{M'l + M''(l-a)}{M}\frac{d^2\cos\alpha}{dt^2} = \frac{W-P}{M}.$$

Hence,

$$\frac{M'l + M''(l-a)}{M}\frac{f'}{l} = \frac{W-P}{M}.$$

If P is constant, then f', which is the only possible variable in this equation, must also be constant. This proves the first part.

If f be the value of f' when $P=0$, we have